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Global singularity structure of weak solutions to 3-D semilinear dispersive wave equations with discontinuous initial data[☆]

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Abstract

We study the global singularity structure of solutions to 3-D semilinear wave equations with discontinuous initial data. More precisely, using Strichartz' inequality we show that the solutions stay conormal after nonlinear interaction if the Cauchy data are conormal along a circle.

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This note is devoted to the global singularity structure of solutions to the following 3-D semilinear wave equation with discontinuous initial data

$$\begin{cases} \square u + g(u) = 0, \\ u(0, x) = 0, \\ \partial_t u(0, x) = u_1(x), \end{cases} \quad (1)$$

where $\square u = \partial_t^2 u - \sum_{k=1}^3 \partial_k^2 u$, $u_1(x) = \varphi(x)$ when $|x| \leq 1$, $u_1(x) = 0$ when $|x| > 1$, $\varphi(x) \in C^\infty(\overline{B(0, 1)})$, and $x = (x_1, x_2, x_3)$. The nonlinearity $g(u) \in C^\infty(\mathbb{R})$ is assumed

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to satisfy the following assumptions:

$$g(0) = 0, \quad G(u) = \int_0^u g(s) ds \geq 0, \quad (2)$$

$$|g^{(j)}(u)| \leq C_j(1 + |u|)^{p-j}, \quad 1 < p \leq 5. \quad (3)$$

Under conditions (2) and (3), the global existence of weak and smooth solutions to (1) has been intensively studied in literature, [8,6,7,10] and the references therein. For instance, if $u_1(x) \in C^\infty(\mathbb{R}^3)$, the authors in [7,8] proved the global existence of smooth solutions to (1) when $p < 5$ and $p = 5$, respectively. Besides, if $u_1(x) \in L^2(\mathbb{R}^3)$, the authors in [6] proved the global existence and uniqueness of weak solutions to (1) in the space $u \in C([0, \infty), H^1(\mathbb{R}^3)) \cap C^1([0, \infty), L^2(\mathbb{R}^3))$ when $1 < p < 5$. Shatah and Struwe [10] then obtained the global existence and uniqueness in the space $u \in C([0, \infty), H^1(\mathbb{R}^3)) \cap C^1([0, \infty), L^2(\mathbb{R}^3)) \cap L^5([0, \infty), L^{10}(\mathbb{R}^3))$ when $p = 5$.

Our main purpose is to study the influence of the nonlinear term $g(u)$ on the properties of the weak solution u to (1), in particular, to give a precise description of the global singularity structure of u when the initial data $u_1(x)$ are discontinuous.

The main result is:

Theorem. *Under conditions (2) and (3), the global weak solution $u(t, x)$ to (1) belongs to $C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) \setminus (\Sigma_1 \cup \Sigma_2))$, where $\Sigma_{1,2} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 : (t \pm 1)^2 = \sum_{k=1}^3 x_k^2\}$.*

Notice that the surface Σ_2 has a singular point $(1, 0, 0, 0)$. In general, the solution u will be singular at this point. Beals [2], showed that singularities of solutions to nonlinear wave equations at such a point can cause a very complicated singularity structure of the solutions for later times. More precisely, he proved that there are a C^∞ function $\beta(t, x)$ and a solution $u \in H_{\text{loc}}^s(\mathbb{R}^+ \times \mathbb{R}^3)$, $s > \frac{3}{2}$, to the equation $\square u + \beta(t, x)u^3 = 0$ such that the initial data $u(0, x)$, $\partial_t u(0, x)$ have compact support and are C^∞ with the exception of the point at $x = 0$, while the singularities of u fill the whole solid light cone $\{(t, x) : |x| \leq t\}$. For problem (1), however, thanks to the special property of the discontinuous initial data to be conormal with respect to the sphere $|x| = 1$ (for the definition of being conormal, see [3] or [4]), with the help of Strichartz' inequality we can prove that the weak solution u to (1) is also globally conormal with respect to both the surfaces Σ_1 and Σ_2 . Consequently, $u \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) \setminus (\Sigma_1 \cup \Sigma_2))$, and the nonlinear term $g(u)$ does not bring about the nonlinear influence on the singularity structure of u .

To prove the Theorem, we will make use of the commutator technique introduced in [4]. To do so, we are required to know a basis for the C^∞ vector fields which are tangent to both Σ_1 and Σ_2 .

Lemma 1. *A basis for the C^∞ vector fields simultaneously tangent to Σ_1 and Σ_2 is given by*

$$V_1 = (t^2 + x_1^2 + x_2^2 + x_3^2 - 1)\partial_t + 2tx_1\partial_1 + 2tx_2\partial_2 + 2tx_3\partial_3,$$

$$V_2 = 2tx_1\partial_t + (t^2 + x_1^2 - 1)\partial_1 + x_1x_2\partial_2 + x_1x_3\partial_3,$$

$$V_3 = 2tx_2\partial_t + x_1x_2\partial_1 + (t^2 + x_2^2 - 1)\partial_2 + x_2x_3\partial_3,$$

$$V_4 = 2tx_3\partial_t + x_1x_3\partial_1 + x_2x_3\partial_2 + (t^2 + x_3^2 - 1)\partial_3,$$

$$V_5 = x_1\partial_2 - x_2\partial_1,$$

$$V_6 = x_1\partial_3 - x_3\partial_1,$$

$$V_7 = x_2\partial_3 - x_3\partial_2.$$

Proof. As is well known, a basis for the C^∞ vector fields tangent to Σ_1 is

$$M_0 = (t + 1)\partial_t + x_1\partial_1 + x_2\partial_2 + x_3\partial_3,$$

$$M_1 = (t + 1)\partial_1 + x_1\partial_t,$$

$$M_2 = (t + 1)\partial_2 + x_2\partial_t,$$

$$M_3 = (t + 1)\partial_3 + x_3\partial_t,$$

$$M_4 = x_1\partial_2 - x_2\partial_1,$$

$$M_5 = x_1\partial_3 - x_3\partial_1,$$

$$M_6 = x_2\partial_3 - x_3\partial_2.$$

Obviously, M_4, M_5 and M_6 are also tangent to Σ_2 .

Suppose that the C^∞ vector field V is tangent to Σ_1 and Σ_2 . Then

$$\begin{aligned} V = \sum_{i=0}^6 a_i(t, x) M_i &= (a_0(t + 1) + a_1x_1 + a_2x_2 + a_3x_3)\partial_t + (a_0x_1 + a_1(t + 1))\partial_1 \\ &+ (a_0x_2 + a_2(t + 1))\partial_2 + (a_0x_3 + a_3(t + 1))\partial_3 + \sum_{i=4}^6 a_i M_i, \end{aligned}$$

where $a_i(t, x)$ ($i = 0, 1, 2, 3$) are appropriate C^∞ functions to be determined.

For V is tangent to Σ_2 , from the Malgrange preparation theorem we infer that there is a C^∞ function $d(t, x)$ such that

$$V \left((t - 1)^2 - \sum_{i=1}^3 x_i^2 \right) = 2d \left((t - 1)^2 - \sum_{i=1}^3 x_i^2 \right).$$

We get

$$\begin{aligned} & \left(a_0(t+1) + \sum_{i=1}^3 a_i x_i \right) (t-1) - \sum_{i=1}^3 (a_0 x_i + a_i(t+1)) x_i \\ &= d \left((t-1)^2 - \sum_{i=1}^3 x_i^2 \right). \end{aligned} \quad (4)$$

We now start to determine the functions a_i ($i = 0, 1, 2, 3$).

Setting $x_1 = x_2 = x_3 = 0$ in (4), we get

$$a_0(t, 0, 0, 0)(t+1) = d(t, 0, 0, 0)(t-1).$$

Then there are C^∞ functions $b_i(t, x)$ ($i = 1, 2, 3$) such that

$$a_0(t+1) = d(t-1) + \sum_{i=1}^3 b_i x_i. \quad (5)$$

Substituting (5) into (4) yields

$$\left(\sum_{i=1}^3 (a_i + b_i) x_i \right) (t-1) - \sum_{i=1}^3 (a_0 x_i + a_i(t+1)) x_i = -d \sum_{i=1}^3 x_i^2. \quad (6)$$

Setting $t = 1$, $x_1 = x_2 = 0$ in (6), on the analogy of the analysis made above we conclude that there are C^∞ functions $b_j(t, x)$ ($j = 4, 5, 6$) such that

$$a_0 x_3 + a_3(t+1) = d x_3 + b_4(t-1) + b_5 x_1 + b_6 x_2. \quad (7)$$

Similarly, there are C^∞ functions $b_k(t, x)$ ($k = 7, 8, 9$) and $b_m(t, x)$ ($m = 10, 11, 12$) such that

$$a_0 x_2 + a_2(t+1) = d x_2 + b_7(t-1) + b_8 x_1 + b_9 x_3, \quad (8)$$

$$a_0 x_1 + a_1(t+1) = d x_1 + b_{10}(t-1) + b_{11} x_2 + b_{12} x_3. \quad (9)$$

Then Eq. (6) assumes the form

$$\begin{aligned} & \left(\sum_{i=1}^3 (a_i + b_i) x_i \right) (t-1) - (b_{10}(t-1) + b_{11} x_2 + b_{12} x_3) x_1 \\ & - (b_7(t-1) + b_8 x_2 + b_9 x_3) x_2 - (b_4(t-1) + b_5 x_1 + b_6 x_2) x_3 = 0. \end{aligned} \quad (10)$$

Setting $t = 1$ and $x_3 = 0$ in (10), we get

$$b_{11}(1, x_1, x_2, 0) + b_8(1, x_1, x_2, 0) = 0.$$

Thus, there are C^∞ functions $c_i(t, x)$ ($i = 1, 2$) such that

$$b_{11} = -b_8 + c_1(t-1) + c_2x_3. \quad (11)$$

Similarly, there are C^∞ functions $c_j(t, x)$ ($j = 3, \dots, 8$) such that

$$b_{12} = -b_5 + c_3(t-1) + c_4x_2, \quad (12)$$

$$b_9 = -b_6 + c_5(t-1) + c_6x_1, \quad (13)$$

$$a_3 + b_3 = b_4 + c_7x_1 + c_8x_2. \quad (14)$$

Eq. (10) becomes

$$\begin{aligned} & \left(\sum_{i=1}^2 (a_i + b_i)x_i + (c_7x_1 + c_8x_2)x_3 \right) (t-1) - (b_{10}(t-1) + (c_1(t-1) + c_2x_3)x_2 \\ & + (c_3(t-1) + c_4x_2)x_3)x_1 - (b_7(t-1) + (c_5(t-1) + c_6x_1)x_3)x_2 = 0. \end{aligned} \quad (15)$$

Setting $t = 1$ and $x_1 = 0$, respectively, in (15), we infer that there are C^∞ functions $c_k(t, x)$ ($k = 9, 10$) such that

$$c_2 + c_4 = -c_6 + c_9(t-1), \quad (16)$$

$$b_7 = a_2 + b_2 + c_8x_3 - c_5x_3 - c_{10}x_1. \quad (17)$$

Thus, Eq. (15) simplifies to

$$a_1 + b_1 - b_{10} = -c_{10}x_2 - c_7x_3 + c_1x_2 + c_3x_3 + c_9x_2x_3. \quad (18)$$

Below, we shall derive expressions for a_0, a_1, a_2, a_3 in terms of $b_1, b_2, b_3, b_5, b_6, b_8, c_6, c_7, c_8, c_{10}$, and d . From (5), (7), (8) and (9), we gather

$$a_0 = \frac{d(t-1) + \sum_{i=1}^3 b_i x_i}{t+1}, \quad (19)$$

$$a_1 = \frac{2dx_1}{(t+1)^2} + \frac{b_{10}(t-1) + b_{11}x_2 + b_{12}x_3}{t+1} - \frac{(\sum_{i=1}^3 b_i x_i)x_1}{(t+1)^2}, \quad (20)$$

$$a_2 = \frac{2dx_2}{(t+1)^2} + \frac{b_7(t-1) + b_8x_1 + b_9x_3}{t+1} - \frac{(\sum_{i=1}^3 b_i x_i)x_2}{(t+1)^2}, \quad (21)$$

$$a_3 = \frac{2dx_3}{(t+1)^2} + \frac{b_4(t-1) + b_5x_1 + b_6x_2}{t+1} - \frac{(\sum_{i=1}^3 b_i x_i)x_3}{(t+1)^2}. \quad (22)$$

Substituting (11)–(13) into (20) and (21), we arrive at new expressions for a_1 and a_2 . Inserting these new expressions for a_1 , a_2 and (22) into (14), (17), and (18), we obtain

$$\left\{ \begin{array}{l} \frac{2b_4}{t+1} = \frac{2dx_3}{(t+1)^2} + \frac{b_5x_1 + b_6x_2}{t+1} - \frac{(\sum_{i=1}^3 b_ix_i)x_3}{(t+1)^2} - c_7x_1 - c_8x_2 + b_3, \\ \frac{2b_7}{t+1} = \frac{2dx_2}{(t+1)^2} + \frac{b_8x_1 + (-b_6 + c_5(t-1) + c_6x_1)x_3}{t+1} - \frac{(\sum_{i=1}^3 b_ix_i)x_2}{(t+1)^2} \\ \quad + b_2 + c_8x_3 - c_5x_3 - c_{10}x_1, \\ \frac{2b_{10}}{t+1} = \frac{2dx_1}{(t+1)^2} + \frac{(-b_8 + c_1(t-1) + c_2x_3)x_2 + (-b_5 + c_3(t-1) + c_4x_2)x_3}{t+1} \\ \quad - \frac{(\sum_{i=1}^3 b_ix_i)x_1}{(t+1)^2} + b_1 + (c_{10} - c_1)x_2 + (c_7 - c_3)x_3 - c_9x_2x_3. \end{array} \right. \quad (23)$$

By virtue of (23) and taking advantage of relation (16), we then find the sought expressions for a_1 , a_2 , and a_3 :

$$\left\{ \begin{array}{l} a_1 = \frac{dx_1}{t+1} - \frac{1}{2} \left(b_8x_2 + c_6x_2x_3 + b_5x_3 + \frac{(\sum_{i=1}^3 b_ix_i)x_1}{t+1} \right) \\ \quad + \frac{t-1}{2} (b_1 + c_{10}x_2 + c_7x_3), \\ a_2 = \frac{dx_2}{t+1} + \frac{1}{2} \left(b_8x_1 + (c_6x_1 - b_6)x_3 - \frac{(\sum_{i=1}^3 b_ix_i)x_2}{t+1} \right) \\ \quad + \frac{t-1}{2} (b_2 + c_8x_3 - c_{10}x_1), \\ a_3 = \frac{dx_3}{t+1} + \frac{1}{2} \left(b_5x_1 + b_6x_2 - \frac{(\sum_{i=1}^3 b_ix_i)x_3}{t+1} \right) \\ \quad + \frac{t-1}{2} (b_3 - c_7x_1 - c_8x_2). \end{array} \right. \quad (24)$$

Substituting (19) and (24) into the expression for V , we get

$$\begin{aligned} V = & \frac{d}{t+1} V_1 + \frac{b_1}{2(t+1)} \left\{ x_1((t+1)^2 - x_1^2 - x_2^2 - x_3^2) \partial_t + (t-1)((t+1)^2 - x_1^2) \partial_1 \right. \\ & \left. + (1-t)x_1x_2\partial_2 + (1-t)x_1x_3\partial_3 \right\} + \frac{b_2}{2(t+1)} \left\{ x_2((t+1)^2 - x_1^2 - x_2^2 - x_3^2) \partial_t \right. \\ & \left. + (1-t)x_1x_2\partial_1 + (t-1)((t+1)^2 - x_2^2) \partial_2 + (1-t)x_1x_3\partial_3 \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{b_3}{2(t+1)} \{x_3((t+1)^2 - x_1^2 - x_2^2 - x_3^2)\partial_t + (1-t)x_1x_3\partial_1 + (1-t)x_2x_3\partial_2 \\
& + (t-1)((t+1)^2 - x_3^2)\partial_3\} + \left(a_4 + \frac{b_8(t+1)}{2} + \frac{c_6x_3(t+1)}{2} - \frac{c_{10}(t^2-1)}{2}\right)V_5 \\
& + \left(a_5 + \frac{b_5(t+1)}{2} - \frac{c_7(t^2-1)}{2}\right)V_6 + \left(a_6 + \frac{b_6(t+1)}{2} - \frac{c_8(t^2-1)}{2}\right)V_7.
\end{aligned}$$

Finally, upon noting

$$\begin{aligned}
& x_1((t+1)^2 - x_1^2 - x_2^2 - x_3^2)\partial_t + (t-1)((t+1)^2 - x_1^2)\partial_1 \\
& + (1-t)x_1x_2\partial_2 + (1-t)x_1x_3\partial_3 = (t+1)V_2 - x_1V_1, \\
& x_2((t+1)^2 - x_1^2 - x_2^2 - x_3^2)\partial_t + (1-t)x_1x_2\partial_1 + (t-1)((t+1)^2 - x_2^2)\partial_2 + (1-t)x_2x_3\partial_3 \\
& = (t+1)V_3 - x_2V_1, \\
& x_3((t+1)^2 - x_1^2 - x_2^2 - x_3^2)\partial_t + (1-t)x_1x_3\partial_1 \\
& + (1-t)x_2x_3\partial_2 + (t-1)((t+1)^2 - x_3^2)\partial_3 = (t+1)V_4 - x_3V_1,
\end{aligned}$$

Lemma 1 is completely proved. \square

Let $[A, B] = AB - BA$ denote the commutator. By direct computation, we find:

Lemma 2. *We have*

$$\begin{aligned}
[V_1, V_2] &= -2x_2tV_5 - 2x_3tV_6, \\
[V_1, V_3] &= 2x_1tV_5 - 2x_3tV_7, \\
[V_1, V_4] &= 2x_1tV_6 + 2x_2tV_7, \\
[V_1, V_5] &= [V_1, V_6] = [V_1, V_7] = 0, \\
[V_2, V_3] &= -3(t^2+1)V_5, \quad [V_2, V_4] = 3(t^2+1)V_6, \\
[V_2, V_5] &= V_3, \quad [V_2, V_6] = V_4, \quad [V_2, V_7] = 0, \\
[V_3, V_4] &= -3(t^2+1)V_7, \quad [V_3, V_5] = -V_2, \\
[V_3, V_6] &= 0, \quad [V_3, V_7] = V_4,
\end{aligned}$$

$$\begin{aligned}
[\square, V_1] &= 4t\square - 4\partial_t, \\
[\square, V_2] &= 4x_1\square + 2\partial_2 V_5 + 2\partial_3 V_6 + 4\partial_1, \\
[\square, V_3] &= 4x_2\square - 2\partial_1 V_5 + 2\partial_3 V_7 + 4\partial_2, \\
[\square, V_4] &= 4x_3\square - 2\partial_1 V_6 - 2\partial_2 V_7 + 4\partial_3, \\
[\square, V_5] &= [\square, V_6] = [\square, V_7] = 0.
\end{aligned}$$

From Lemma 2, one easily derives the following result:

Corollary. Let $\{V^k u\} = \{V_{l_1}^{k_1} V_{l_2}^{k_2} \cdots V_{l_j}^{k_j} u : k_1 + \cdots + k_j = |k|, k_1, \dots, k_j \in \mathbb{N}_0\}$, where $V_{l_1}, \dots, V_{l_j} \in \{V_1, \dots, V_7\}$. If u is a solution to (1), then $\{V^k u\}$ satisfies the following system of equations:

$$\begin{aligned}
&\square V^k u + \sum_{j=1}^3 c_{kj}(t, x) \partial_j V^k u + f_k(t, x, u) V^k u \\
&= \sum_{|l_1| + \cdots + |l_j| \leq |k|-1} a_{kl_1 \dots l_j}(x, t, u) V^{l_1} u \cdots V^{l_j} u + \sum_{\substack{1 \leq j \leq 3 \\ |l| \leq |k|-1}} b_{klj} \partial_j V^l u, \quad (25)
\end{aligned}$$

where $c_{kj}(t, x)$, $f_k(t, x, u)$, and $a_{kl_1 \dots l_j}(x, t, u)$ are C^∞ functions, b_{klj} are constants.

Next, we determine the initial data for the $V^k u$.

Lemma 3. For each k , we have

$$\begin{cases} V^k u|_{t=0} = g_{1k}(x)(x_1^2 + x_2^2 + x_3^2 - 1)H(x_1^2 + x_2^2 + x_3^2 - 1) + g_{2k}(x), \\ \partial_t V^k u|_{t=0} = q_{1k}(x)H(x_1^2 + x_2^2 + x_3^2 - 1) + q_{2k}(x), \end{cases} \quad (26)$$

where $H(s)$ is the Heaviside function, and $g_{1k}(x)$, $g_{2k}(x)$, $q_{1k}(x)$, $q_{2k}(x)$ are C^∞ on \mathbb{R}^3 .

Proof. For $|k| = 0$, (26) obviously holds.

We now assume that (26) holds for some $|k|$. Then we want to show that (26) also holds when $|k|$ is replaced with $|k| + 1$.

Let $N_1 = (x_1^2 - 1)\partial_1 + x_1 x_2 \partial_2 + x_1 x_3 \partial_3$, $N_2 = x_1 x_2 \partial_1 + (x_2^2 - 1)\partial_2 + x_2 x_3 \partial_3$, and $N_3 = x_1 x_3 \partial_1 + x_2 x_3 \partial_2 + (x_3^2 - 1)\partial_3$. Then $\{N_1, N_2, N_3, V_5, V_6, V_7\}$ constitutes a basis for the C^∞ vector fields tangent to the circle $x_1^2 + x_2^2 + x_3^2 = 1$. A direct

computation gives

$$\begin{aligned} V_1 V^k u|_{t=0} &= (x_1^2 + x_2^2 + x_3^2 - 1) \partial_t V^k u|_{t=0} \\ &= q_{1k}(x)(x_1^2 + x_2^2 + x_3^2 - 1) H(x_1^2 + x_2^2 + x_3^2 - 1) \\ &\quad + q_{2k}(x)(x_1^2 + x_2^2 + x_3^2 - 1), \end{aligned}$$

$$\begin{aligned} V_{i+1} V^k u|_{t=0} &= (N_i g_{1k}(x) + 2x_i g_{1k}(x))(x_1^2 + x_2^2 + x_3^2 - 1) H(x_1^2 + x_2^2 + x_3^2 - 1) \\ &\quad + N_i g_{2k}(x), \quad i = 1, 2, 3, \end{aligned}$$

$$\begin{aligned} V_{j+4} V^k u|_{t=0} &= V_{j+4} g_{1k}(x)(x_1^2 + x_2^2 + x_3^2 - 1) H(x_1^2 + x_2^2 + x_3^2 - 1) \\ &\quad + V_{j+4} g_{2k}(x), \quad j = 1, 2, 3, \end{aligned}$$

$$\begin{aligned} V_1 \partial_t V^k u|_{t=0} &= (x_1^2 + x_2^2 + x_3^2 - 1) \partial_t^2 V^k u|_{t=0} \\ &= (x_1^2 + x_2^2 + x_3^2 - 1) \left(\sum_{k=1}^3 \partial_k^2 V^k u - \sum_{j=1}^3 c_{kj}(t, x) \partial_j V^k u \right. \\ &\quad \left. - f_k(t, x, u) V^k u + \sum_{|l_1| + \dots + |l_j| \leq |k| - 1} a_{kl_1 \dots l_j}(t, x, u) V^{l_1} u \dots V^{l_j} u \right. \\ &\quad \left. + \sum_{\substack{1 \leq j \leq 3 \\ |m| \leq |k| - 1}} b_{kmj} \partial_j V^m u \right) \Big|_{t=0} \\ &= g_{1,k+1}(x)(x_1^2 + x_2^2 + x_3^2 - 1) H(x_1^2 + x_2^2 + x_3^2 - 1) + g_{2,k+1}(x), \end{aligned}$$

$$V_{i+1} \partial_t V^k u|_{t=0} = N_i q_{1k}(x) H(x_1^2 + x_2^2 + x_3^2 - 1) + N_i q_{2k}(x), \quad i = 1, 2, 3,$$

$$V_{j+1} \partial_t V^k u|_{t=0} = V_{j+4} q_{1k}(x) H(x_1^2 + x_2^2 + x_3^2 - 1) + V_{j+4} q_{2k}(x), \quad j = 1, 2, 3.$$

By the principle of induction and a straightforward computation, Lemma 3 is then proved. \square

To solve (25) with the initial data (26), we have to show that the weak solution $u(t, x)$ to (1) belongs to $L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3)$. Let us recall Strichartz' inequality from [7].

Lemma 4. *There is a constant $C > 0$ such that, for each $T > 0$,*

$$\|u\|_{L^q(0,T;L^{3r}(\mathbb{R}^3))} \leq C \left(\|\square u\|_{L^1(0,T;L^2(\mathbb{R}^3))} + \|\nabla_x u(0,x)\|_{L^2(\mathbb{R}^3)} + \|\partial_t u(0,x)\|_{L^2(\mathbb{R}^3)} \right), \quad (27)$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $r \in [2, \infty)$.

By virtue of estimate (27), we arrive at (see [7, Lemma 3.3]):

Lemma 5. *For all $T > 0$, $r \in [2, \infty)$, $M > 0$, there is a constant $C(T, r, M) > 0$ such that the solution u to (1) with $p < 5$ satisfies*

$$\|u\|_{L^q(0,T;L^{3r}(\mathbb{R}^3))} \leq C(T, r, M), \quad (28)$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $r \in [2, \infty)$, and $\|u_1\|_{L^2(\mathbb{R}^3)} \leq M$.

Based on Lemmas 4 and 5, we can now establish an L_{loc}^∞ bound for solutions u to (1) when $p < 5$.

Lemma 6. *Let $1 < p < 5$. Then the weak solution u to (1) satisfies*

$$u(t, x) \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3).$$

Proof. Suppose that $v(t, x)$ is the solution of the following linear wave equation:

$$\begin{cases} \square v = 0, \\ v(0, x) = 0, \\ \partial_t v(0, x) = u_1(x). \end{cases}$$

By Kirchhoff's formula,

$$v(t, x) = \frac{1}{4\pi t} \int_{|x-y|=t} u_1(y) dS_y = \frac{t}{4\pi} \int_{S^2} u_1(x + t\omega) d\omega.$$

Hence, $v(t, x) \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3)$. By virtue of the energy inequality, we additionally have

$$v(t, x) \in C([0, \infty), H_{\text{comp}}^{s+1}(\mathbb{R}^3)) \cap C^1([0, \infty), H_{\text{comp}}^s(\mathbb{R}^3)) \quad \text{for any } s < \frac{1}{2}.$$

Set $w = u - v$. Then w satisfies

$$\begin{cases} \square w + g(w + v) = 0, \\ w(0, x) = 0, \\ \partial_t w(0, x) = 0. \end{cases} \quad (29)$$

To show $u \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3)$, it suffices to show that $w \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3)$.

From (29), one deduces

$$\begin{cases} \square \partial_x w + g'(w + v) \partial_x(w + v) = 0, \\ \partial_x w(0, x) = 0, \\ \partial_t \partial_x w(0, x) = 0. \end{cases} \quad (30)$$

What we want to show is $g'(w + v) \partial_x(w + v) \in L_{\text{loc}}^1([0, \infty); L^2(\mathbb{R}^3))$. If this holds, then $\partial_x^2 w \in L_{\text{loc}}^\infty([0, \infty); L^2(\mathbb{R}^3))$ and $w \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3)$ by Sobolev's embedding theorem.

In fact, using (28) and choosing $q = 5$, $r = \frac{10}{3}$ in Lemma 4, we have

$$\begin{aligned} \|\partial_x w\|_{L^5(0, T; L^{10})} &\leq C \|g'(w + v) \partial_x(w + v)\|_{L^1(0, T; L^2)} \\ &\leq C_{T, M} (\|\partial_x w\|_{L^\infty(0, T; L^2)} + \|\partial_x v\|_{L^\infty(0, T; L^2)} \\ &\quad + \| |w|^{p-1} \partial_x w \|_{L^1(0, T; L^2)} + \| |w|^{p-1} \partial_x v \|_{L^1(0, T; L^2)}) \\ &\leq C_{T, M} (1 + \|w\|_{L^{\frac{5(p-1)}{4+\theta}}(0, T; L^{\frac{5(p-1)}{2(1-\theta)}}})^{p-1} \|\partial_x w\|_{L^\infty(0, T; L^2)}^\theta \|\partial_x v\|_{L^5(0, T; L^{10})}^{1-\theta} \\ &\quad + \| |w|^{p-1} \partial_x v \|_{L^1(0, T; L^2)}), \end{aligned}$$

where $\theta = \frac{5-p}{8}$, and $C_{T, M} > 0$ is a generic constant depending only on $T > 0$ and $M > 0$, where $\|u_1\|_{L^2(\mathbb{R}^3)} \leq M$. Taking into account that $\frac{5(p-1)}{4+\theta} < 5$, $\frac{5(p-1)}{2(1-\theta)} < 10$, and $\|w\|_{L^5(0, T; L^{10})} \leq C_{T, M}$ in view of Lemma 5, we further obtain

$$\|\partial_x w\|_{L^5(0, T; L^{10})} \leq C_{T, M} (1 + \|\partial_x w\|_{L^5(0, T; L^{10})}^{1-\theta} + \| |w|^{p-1} \partial_x v \|_{L^1(0, T; L^2)}). \quad (31)$$

To estimate the remaining term $\| |w|^{p-1} \partial_x v \|_{L^1(0, T; L^2)}$, we now distinguish three cases:

Case 1: $1 < p < 2$. By Sobolev's embedding theorem, $\partial_x v \in L^\infty(0, T; L^{3-\varepsilon_0})$ for any $\varepsilon_0 > 0$. Since $|w|^{p-1} \in L^\infty(0, T; L^{\frac{6}{p-1}})$, upon choosing $\varepsilon_0 = \frac{6-3p}{4-p} < 1$ we obtain

$$\| |w|^{p-1} \partial_x v \|_{L^1(0, T; L^2)} \leq C_{T, M} \|\partial_x v\|_{L^\infty(0, T; L^{3-\varepsilon_0})} \| |w|^{p-1} \|_{L^\infty(0, T; L^{\frac{6}{p-1}})} \leq C_{T, M}$$

by Hölder's inequality. Hence, for $1 < p < 2$, (31) implies

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \leq C_{T,M}(1 + \|\partial_x w\|_{L^5(0,T;L^{10})}^{1-\theta}). \quad (32)$$

We deduce $\|\partial_x w\|_{L^5(0,T;L^{10})} \leq C_{T,M}$ from (32), i.e., $g'(w+v)\partial_x(w+v) \in L^1(0,T;L^2)$ and $w \in L^\infty((0,T) \times \mathbb{R}^3)$.

Case 2: $2 \leq p < 4$. By Hölder's inequality, we have

$$\begin{aligned} & \| |w|^{p-1} \partial_x v \|_{L^1(0,T;L^2)} \\ & \leq \|1\|_{L^{p_1}(0,T;L^{q_1}(B(0,1+T)))} \| |w|^{p-1} \|_{L^{(p-1)p_2}(0,T;L^{(p-1)q_2})} \|\partial_x v\|_{L^\infty(0,T;L^{3-\varepsilon_0})}, \end{aligned}$$

where $p_1, p_2, q_1, q_2 \geq 1$ satisfy

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} &= 1, \quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{3-\varepsilon_0} = \frac{1}{2}, \\ \frac{1}{(p-1)p_2} + \frac{3}{(p-1)q_2} &= \frac{1}{2} \quad (p-1)q_2 \geq 6. \end{aligned}$$

Choose $0 < \varepsilon_0 < \min\{1, 4-p\}$ and set $p_1 = \frac{2}{4-p-\varepsilon_0}$, $q_1 = \frac{6(3-\varepsilon_0)}{\varepsilon_0(1-\varepsilon_0)}$, $p_2 = \frac{2}{p-2+\varepsilon_0}$, and $q_2 = \frac{6}{1-\varepsilon_0}$ to satisfy the above requirements. Hence, in view of Lemma 5, we obtain $\| |w|^{p-1} \partial_x v \|_{L^1(0,T;L^2)} \leq C_{T,M}$. Similar to Case 1, we then deduce $w \in L^\infty((0,T) \times \mathbb{R}^3)$.

Case 3: $4 \leq p < 5$. Choose $\varepsilon_0 > 0$ sufficiently small so that $p-2+\varepsilon_0 < 3$ and $\frac{(p-2+\varepsilon_0)(3-\varepsilon_0)}{1-\varepsilon_0} < 9$. By Hölder's inequality and Sobolev's embedding theorem,

$$\begin{aligned} & \| |w|^{p-1} \partial_x v \|_{L^1(0,T;L^2)} \\ & = \| |w|^{p-2+\varepsilon_0} |w|^{1-\varepsilon_0} \partial_x v \|_{L^1(0,T;L^2)} \\ & \leq \| |w|^{p-2+\varepsilon_0} \|_{L^{\frac{2(p-2+\varepsilon_0)(3-\varepsilon_0)}{1-\varepsilon_0}}(0,T;L^{\frac{2(p-2+\varepsilon_0)(3-\varepsilon_0)}{1-\varepsilon_0}})} \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{1-\varepsilon_0} \|\partial_x v\|_{L^\infty(0,T;L^{3-\varepsilon_0})}. \end{aligned}$$

By Lemma 5 (with $q = 3$, $r = 6$), we get

$$\| |w|^{p-1} \partial_x v \|_{L^1(0,T;L^2)} \leq C_{T,M} \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{1-\varepsilon_0}. \quad (34)$$

Substituting (34) into (31) gives

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \leq C_{T,M} \left(1 + \|\partial_x w\|_{L^5(0,T;L^{10})}^\theta + \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{1-\varepsilon_0} \right).$$

Consider the function $h(s) = s - C_0 - C_{T,M}s^\theta$, where $C_0 = C_{T,M}(1 + \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{1-\varepsilon_0})$. Thereby, we can assume that $C_{T,M} > 1$. Obviously, $h(0) = -C_0 < 0$ and

$$\begin{aligned} h(C_0 + C_0 C_{T,M}^m) &= C_0^\theta C_{T,M} (C_0^{1-\theta} C_{T,M}^{m-1} - (1 + C_{T,M}^m)^\theta) \\ &> C_0^\theta C_{T,M} (C_{T,M}^{m-1} - (1 + C_{T,M}^m)^\theta) > 0 \end{aligned}$$

for $m > 0$ large enough. Hence,

$$\|\partial_x w\|_{L^5(0,T;L^{10})} \leq C_0(1 + C_{T,M}^m) \leq C_{T,M} \left(1 + \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{1-\varepsilon_0}\right).$$

Upon applying the energy estimate to (30), we infer

$$\begin{aligned} \|\partial_x^2 w\|_{L^\infty(0,T;L^2)} &\leq C_{T,M} \|g'(w+v)\partial_x(w+v)\|_{L^1(0,T;L^2)} \\ &\leq C_{T,M} \left(1 + \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{1-\varepsilon_0} + \|\partial_x^2 w\|_{L^\infty(0,T;L^2)}^{(1-\varepsilon_0)\theta}\right). \end{aligned}$$

Hence, $\|\partial_x^2 w\|_{L^\infty(0,T;L^2)} \leq C_{T,M}$, and we find $w \in L^\infty(0, T; H^2(\mathbb{R}^3)) \subset L^\infty((0, T) \times \mathbb{R}^3)$.

Because $T > 0$ is arbitrary, Lemma 6 is proved. \square

Now, we establish the same result when $p = 5$.

Lemma 7. For $p = 5$, the weak solution $u \in C([0, \infty), H^1(\mathbb{R}^3)) \cap C^1([0, \infty), L^2(\mathbb{R}^3)) \cap L^5([0, \infty), L^{10}(\mathbb{R}^3))$ to (1) satisfies

$$u(t, x) \in L_{\text{loc}}^\infty([0, \infty) \times \mathbb{R}^3).$$

Remark. In case $p = 5$, (1) has a unique global weak solution $u \in C([0, \infty), H^1(\mathbb{R}^3)) \cap C^1([0, \infty), L^2(\mathbb{R}^3)) \cap L_{\text{loc}}^5([0, \infty), L^{10}(\mathbb{R}^3))$ by [10]. Moreover, $u \in L^5((0, \infty), L^{10}(\mathbb{R}^3))$ by [1].

Proof. One easily gets that there is a time $T^* > 0$ such that Eq. (29) has a local solution $w(t, x) \in C([0, T^*), H_{\text{comp}}^{s+2}(\mathbb{R}^3)) \cap C^1([0, T^*), H_{\text{comp}}^{s+1}(\mathbb{R}^3))$ for any $0 < s < \frac{1}{2}$. Thus, $w(t, x) \in L_{\text{loc}}^5([0, T^*), L^{10}(\mathbb{R}^3))$. By the preceding remark, $w(t, x) \in L^5((0, T^*), L^{10}(\mathbb{R}^3))$.

We want to show that $w(t, x) \in L^\infty((0, T^*) \times \mathbb{R}^3)$.

Applying the pseudodifferential operator $A^{s+\frac{1}{2}}$ with symbol $(1 + |\xi|^2)^{\frac{1}{2}(s+\frac{1}{2})}$ to both sides of (29), we find

$$\begin{aligned} \square A^{s+\frac{1}{2}} w + A^{s+\frac{1}{2}} g(w+v) &= 0, \\ A^{s+\frac{1}{2}} w(0, x) &= 0, \\ \partial_t A^{s+\frac{1}{2}} w(0, x) &= 0. \end{aligned} \tag{35}$$

Choosing $t_0 < s_0 < T^*$, by virtue of Lemma 4 we have

$$\begin{aligned} & \|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})} \\ & \leq C(\|\partial_x A^{s+\frac{1}{2}}w(t_0, \cdot)\|_{L^2} + \|\partial_t A^{s+\frac{1}{2}}w(t_0, \cdot)\|_{L^2} \\ & \quad + \|A^{s+\frac{1}{2}}g(w+v)\|_{L^1(t_0, s_0; L^2)}) \leq C(t_0) + C\|(w+v)^5\|_{L^1(t_0, s_0; H^{s+\frac{1}{2}})}. \end{aligned}$$

For $\mu > 0, 1 < p < \infty$, the following inequality holds (see [5]):

$$\|A^\mu(f_1 f_2)\|_{L^p(\mathbb{R}^n)} \leq C(\|f_1\|_{L^{q_1}(\mathbb{R}^n)} \|A^\mu f_2\|_{L^{q_2}(\mathbb{R}^n)} + \|f_2\|_{L^{r_1}(\mathbb{R}^n)} \|A^\mu f_1\|_{L^{r_2}(\mathbb{R}^n)})$$

provided that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}$, $q_1, r_1 \in (1, \infty]$, $q_2, r_2 \in (0, \infty)$.

Thus, we obtain, for $\|u_1\|_{L^2} \leq M$,

$$\begin{aligned} & \|w^5(t, \cdot)\|_{H^{s+\frac{1}{2}}} \\ & \leq C(\|w(t, \cdot)\|_{L^{10}} \|A^{s+\frac{1}{2}}(w^4)(t, \cdot)\|_{L^{\frac{5}{2}}} + \|w^4(t, \cdot)\|_{L^{\frac{5}{2}}} \|A^{s+\frac{1}{2}}w(t, \cdot)\|_{L^{10}}) \\ & \leq C\|w(t, \cdot)\|_{L^{10}}^4 \|A^{s+\frac{1}{2}}w(t, \cdot)\|_{L^{10}}, \|w^4 v(t, \cdot)\|_{H^{s+\frac{1}{2}}} \\ & \leq C(\|w^4(t, \cdot)\|_{L^6} \|A^{s+\frac{1}{2}}v(t, \cdot)\|_{L^3} + \|v(t, \cdot)\|_{L^\infty} \|A^{s+\frac{1}{2}}(w^4)(t, \cdot)\|_{L^2}) \\ & \leq C_{T^*, M}(\|w(t, \cdot)\|_{L^{18}}^3 \|A^{s+\frac{3}{2}}w(t, \cdot)\|_{L^2} + \|A^{s+\frac{1}{2}}(w^4)(t, \cdot)\|_{L^3}) \\ & \leq C_{T^*, M}\|w(t, \cdot)\|_{L^{18}}^3 \|A^{s+\frac{3}{2}}w(t, \cdot)\|_{L^2}, \end{aligned}$$

$$\begin{aligned} & \|w^3 v^2(t, \cdot)\|_{H^{s+\frac{1}{2}}} \\ & \leq C(\|w^3(t, \cdot)\|_{L^6} \|A^{s+\frac{1}{2}}v^2(t, \cdot)\|_{L^3} + \|v^2(t, \cdot)\|_{L^\infty} \|A^{s+\frac{1}{2}}(w^3)(t, \cdot)\|_{L^2}) \\ & \leq C_{T^*, M}(\|w(t, \cdot)\|_{L^{18}}^3 \|v^2(t, \cdot)\|_{H^{s+1}} \\ & \quad + \|v(t, \cdot)\|_{L^\infty}^2 \|w(t, \cdot)\|_{L^{10}}^2 \|A^{s+\frac{1}{2}}w(t, \cdot)\|_{L^{10}}), \end{aligned}$$

$$\begin{aligned} & \|w^2 v^3(t, \cdot)\|_{H^{s+\frac{1}{2}}} \\ & \leq C(\|w^2(t, \cdot)\|_{L^6} \|A^{s+\frac{1}{2}}v^3(t, \cdot)\|_{L^3} + \|v^3(t, \cdot)\|_{L^\infty} \|A^{s+\frac{1}{2}}(w^2)(t, \cdot)\|_{L^2}) \end{aligned}$$

$$\begin{aligned} &\leq C_{T^*,M}(\|w(t, \cdot)\|_{L^{12}}^2 \|v^3(t, \cdot)\|_{H^{s+1}} \\ &\quad + \|v(t, \cdot)\|_{L^\infty}^3 \|w(t, \cdot)\|_{L^{10}} \|A^{s+\frac{1}{2}}w(t, \cdot)\|_{L^{10}}), \end{aligned}$$

$$\|wv^4(t, \cdot)\|_{H^{s+\frac{1}{2}}} \leq C(\|w(t, \cdot)\|_{L^6} \|v^4(t, \cdot)\|_{H^{s+1}} + \|v(t, \cdot)\|_{L^\infty}^4 \|A^{s+\frac{1}{2}}w(t, \cdot)\|_{L^2}).$$

Hence, we have

$$\begin{aligned} &\|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})} \\ &\leq C_{T^*,M}(1 + \|w\|_{L^5(t_0, s_0; L^{10})}^4 \|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})} \\ &\quad + \|w\|_{L^3(t_0, s_0; L^{18})}^3 \|A^{s+\frac{3}{2}}w\|_{L^\infty(t_0, s_0; L^2)} + \|w\|_{L^5(t_0, s_0; L^{10})}^2 \|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})} \\ &\quad + \|w\|_{L^5(t_0, s_0; L^{10})} \|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})} + (s_0 - t_0)^{\frac{4}{3}} \|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})}). \end{aligned} \quad (36)$$

Since $w \in L^5((0, T^*), L^{10})$, we have $w \in L^3([0, T^*), L^{18})$ by virtue of Lemma 4. When t_0 is sufficiently close to T^* , we infer from (36)

$$\|A^{s+\frac{1}{2}}w\|_{L^5(t_0, s_0; L^{10})} \leq C_{T^*,M}(1 + \|w\|_{L^3(t_0, s_0; L^{18})}^3 \|A^{s+\frac{3}{2}}w\|_{L^\infty(t_0, s_0; L^2)}).$$

Applying the energy estimate to (35), we get

$$\begin{aligned} &\|\partial_t A^{s+\frac{1}{2}}w\|_{L^\infty(t_0, s_0; L^2)} + \|A^{s+\frac{3}{2}}w\|_{L^\infty(t_0, s_0; L^2)} \\ &\leq C_{T^*,M} \left(1 + \|w\|_{L^3(t_0, s_0; L^{18})}^3 \|A^{s+\frac{3}{2}}w\|_{L^\infty(t_0, s_0; L^2)} \right). \end{aligned}$$

If t_0 is sufficiently close to T^* so that $C_{T^*,M} \|w\|_{L^3(t_0, s_0; L^{18})}^3 < \frac{1}{2}$, then we have

$$\|\partial_t A^{s+\frac{1}{2}}w\|_{L^\infty(t_0, s_0; L^2)} \leq C_{T^*,M}, \quad \|A^{s+\frac{3}{2}}w\|_{L^\infty(t_0, s_0; L^2)} \leq C_{T^*,M}.$$

More precisely,

$$\|\partial_t A^{s+\frac{1}{2}}w\|_{C([t_0, T^*), L^2)} \leq C_{T^*,M}, \quad \|A^{s+\frac{3}{2}}w\|_{C([t_0, T^*), L^2)} \leq C_{T^*,M}.$$

Using Eq. (35) once again, we find

$$\|\partial_t A^{s+1} w\|_{C([t_0, T^*], L^2)} \leq C_{T^*, M}, \quad \|A^{s+2} w\|_{C([t_0, T^*], L^2)} \leq C_{T^*, M}.$$

We obtain $w \in C([0, T^*], H^{s+2}(\mathbb{R}^3)) \cap C^1([0, T^*], H^{s+1}(\mathbb{R}^3))$ by Arzela–Ascoli’s Theorem. Furthermore, for some $\delta > 0$, we can extend this solution to the time interval $[0, T^* + \delta]$ utilizing the local existence of solutions to (35).

Continuing this way, the proof of Lemma 7 is completed. \square

Proof of the Theorem. We solve Eq. (25) with the initial data (26). To prove the Theorem, we need to show that

$$V^k u \in C([0, \infty), L^2(\mathbb{R}^3)) \quad (37)$$

for all k .

For $|k| = 0$, (37) obviously holds. Now suppose that (37) holds when $|l| \leq |k| - 1$. We shall then prove this result for k .

Since $u \in L^\infty([0, T] \times \mathbb{R}^3)$ for $T > 0$ according to Lemmas 6 and 7, respectively, the Gagliardo–Nirenberg inequality (see [9]) when applied to (25) gives

$$\sum_{|l_1| + \dots + |l_j| \leq |k| - 1} a_{kl_1 \dots l_j}(x, t, u) V^{l_1} u \dots V^{l_j} u \in C([0, T], L^2(\mathbb{R}^3)).$$

Hence, by the standard energy estimate and the regularity of initial data as stated in Lemma 3, we get

$$V^k u \in C([0, T], L^2(\mathbb{R}^3)) \quad \text{for any } T > 0.$$

Therefore, (37) holds and the Theorem is proved. \square

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